

Phase dynamics in a binary-collisions atom laser scheme

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Various aspects of the phase dynamics of an atom laser scheme based on binary collisions are investigated. Analytical estimates of the influence of elastic atom-atom collisions on the laser linewidth are given, and linewidths achievable in a recently proposed atom laser scheme [Phys. Rev. A **56**, 2989 (1997)] are evaluated explicitly. The extent to which a relative phase can be established between two interfering atom lasers, as well as the properties of that phase, are also investigated.

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I. INTRODUCTION

The recent advances in the generation and manipulation of samples of ultracold atoms have stimulated research aimed at the development of sources of coherent atomic beams. Recent MIT experiments [1,2] can be considered as the first demonstration of a pulsed atom laser. An important further goal is to achieve a cw system. Several theoretical proposals to realize such a device have already been made. They can be separated into two categories: in the first one, the population of the atom laser mode results from optical cooling-like processes such that electronically excited atoms undergo a transition to the atom laser mode via spontaneous emission [3–7]. Proposals in the second category [8–12] rely on binary collisions between ground state atoms fed into an intermediary level of the atom resonator. These collisions are such that one of the atoms undergoes a transition into the laser mode and the other into a heavily damped mode. This latter atom rapidly escapes the cavity, thereby providing the irreversibility of the pumping process.

Amongst of the most important characteristics of a laser are the coherence properties of its output. In the context of cw atom lasers, this aspect has been addressed in a number of articles: Refs. [13–16] investigate various output coupling schemes and their effect on the laser linewidth. They show that a narrow linewidth can be achieved by a suitable choice of outcoupling setup. However, the details of the internal dynamics of the atom laser have not been taken into account so far in that work. In contrast, Refs. [9,10] consider the correlation functions for a three-mode model of a binary-collision atom laser, modeling the output mechanism simply as linear damping. These studies show that the elastic atomic collisions which occur in addition to the pumping collisions impose severe limitations on the coherence of the laser output. Finally, estimates of the linewidth of an atom laser based on optical cooling are given in Ref. [3].

The purpose of the present article is to extend the previous work on the phase dynamics of a binary collision atom laser. We describe this system in terms of a three-mode scheme which ignores most of the multitude of atomic cavity levels and takes into account only those modes that are essential for the lasing process, i. e. the pump, laser mode and loss mode. The feeding and loss processes are described by linear coupling to external continua.

Several reasons motivate our interest in the binary collision atom laser and in this particular model. First, and in contrast to the optical pumping approach, evaporative cooling has already demonstrated its value in achieving BEC [17–19]. The binary collision model we consider can be seen as the simplest possible caricature of this mechanism. Second, an interesting proposal has recently been made to overcome some of the problems associated with elastic binary collisions [11,12] in case they result from the dipole-dipole interaction between atoms: By taking advantage of specifically designed optical cavities, it is possible to greatly decrease the rate of elastic collisions between laser mode atoms. It is of interest to examine the resulting influence on the laser linewidth. Third, the binary collision atom lasers can be analyzed within the framework of few-mode models in more detail and more easily than the models based on optical cooling. An additional justification for studying such rudimentary models lies in the fact that they give a good theoretical understanding of the fundamental way an atom laser might generically work. As such, they provide valuable complementary insights to more realistic approaches, e.g. the quantum kinetic theory of BEC which may also be applied to atom lasers [20–22]. Finally, before including the effects of more sophisticated output coupling schemes it is important to gain a thorough understanding of the internal laser dynamics and its impact on the laser linewidth in simple models.

After briefly reviewing the main aspects of binary collisions atom laser schemes in section II, the present paper addresses two main topics: Section III discusses the laser linewidth and its dependence on elastic atom-atom collisions. In previous studies this question was either treated rather briefly [10] or estimates were given [9] the validity of which is not quite clear [23]. Our analysis leads to an analytical approximation for the laser linewidth in the presence of interatomic collisions. To this end a linearized fluctuation analysis of the system master equation is performed after adiabatically eliminating the loss mode. The results of this analysis are compared to quantum Monte Carlo simulations [24,25]. We illustrate these results in the determination of the possible operating regimes of

the atom laser of Refs. [11,12], and demonstrate that in for weak pumping it should in principle be possible to obtain laser linewidths below the natural linewidth of the laser mode.

In section IV we turn to the question of whether a definite relative phase can be established between independent atom lasers. The same question, but for Bose-Einstein condensates, has aroused much interest recently [26–31]. It was shown that for a system of two condensates with a fixed total number of atoms, a relative phase is established by the process of measurement. Our analysis extends this work to the realm of open systems. We show that a relative phase can be established in single runs of the experiment in very much the same way as for Bose condensates. The study of the diffusion properties of this relative phase shows its close connection to the phase of the single atom laser as described in terms of stochastic processes. Finally, a brief summary and conclusions are given in section V.

II. THE BINARY COLLISION ATOM LASER MODEL

The basic principles of operation of the binary collision atom laser are discussed in detail in several publications [8–12]. One considers a resonator for atoms, realized e.g. by optical fields. In order to concentrate on the essential dynamics only three out of the multitude of atomic center-of-mass modes are taken into account explicitly (cf. Fig. 1). Bosonic atoms in their ground electronic state are pumped into an atomic resonator level of “intermediary” energy (mode 1). They then undergo binary collisions which take one of the atoms involved to the tightly bound laser mode 0, whereas the other one is transferred to the heavily damped loss mode 2. This latter atom leaves the resonator quickly, thereby providing the irreversibility of the pumping process. A macroscopic population of the laser mode can built up as soon as the influx of atoms due to pumping compensates for the losses induced by the damping.

In the description of this laser scheme one has to take into account that in addition to the pumping collisions other types of interatomic collisions can also occur. These considerations lead to an ansatz for the atom laser master equation of the form

$$\dot{W} = -i[H_0 + H_{col}, W] + \kappa_0 \mathcal{D}[a_0]W + \kappa_1(N+1)\mathcal{D}[a_1]W + \kappa_1 N \mathcal{D}[a_1^\dagger]W + \kappa_2 \mathcal{D}[a_2]W \quad (1)$$

with $\hbar = 1$. In this equation, we use the second quantized formalism in which each center-of-mass atomic mode is associated with an annihilation operator a_i , and W denotes the atomic density operator¹. The free Hamiltonian

is given by

$$H_0 = \sum_{i=0,1,2} \omega_i a_i^\dagger a_i,$$

ω_i being the mode frequencies. The general form of the collision Hamiltonian can be written as

$$H_{col} = \sum_{i \leq j, k \leq l} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l \quad (2)$$

with V_{ijkl} the matrix elements of the two-body interaction Hamiltonian responsible for the collisions. However, for the present purposes it will be sufficient to restrict our attention to the reduced form

$$H_{col} = V_{0211} a_0^\dagger a_2^\dagger a_1 a_1 + V_{1102} a_1^\dagger a_1^\dagger a_0 a_2 + V_{0000} a_0^\dagger a_0^\dagger a_0 a_0 + V_{0101} a_0^\dagger a_1^\dagger a_0 a_1 + V_{1111} a_1^\dagger a_1^\dagger a_1 a_1 \quad (3)$$

in which (besides the pumping collisions) only those collisions are retained which are expected to have the most significant influence on the phase dynamics. The damping rates of the cavity modes are given by the coefficients κ_i , and the strength of the external pumping of mode 1 is characterized by the parameter N , which is the mean number of atoms to which mode 1 would equilibrate in the absence of collisions. The superoperator \mathcal{D} is of the Lindblad form and is defined by

$$\mathcal{D}[c]P = cPc^\dagger - \frac{1}{2}(c^\dagger cP + Pc^\dagger c) \quad (4)$$

with arbitrary operators c and P .

It has been pointed out [9] that it is useful to study this system in detail although it might be thought of more as a caricature than as an approximation, due to the neglect of most of the modes in the atomic resonator. From this point of view, its main purpose is to give a basic understanding of the laser dynamics, rather than a detailed quantitative description of a realistic setup. However, recent work has added significantly more substance to this model [11,12]: It was shown that in case the binary collisions result from the dipole-dipole interaction and for a quasi one-dimensional modulated cavity, a high mode selectivity for the atomic collisions can be obtained. One particular pumping mode is then coupled only to a small number of other modes. Among the small set of initially populated low-lying modes, a particular one is then expected to be singled out as a result of mode competition. Another benefit of this model is the fact that due to the dipole-dipole selection rules in the resonator, the strength V_{0000} of detrimental collisions can be made small in comparison to V_{0211} . In that proposal, an efficient pumping process is achieved by a time-dependent modulation of the atom cavity. In the spirit of this model we set $\omega_0 + \omega_2 = 2\omega_1$ in the following.

¹Since we consider ground state atoms only, they are fully

described by their center-of-mass quantum numbers.

In order to achieve a sufficiently high degree of irreversibility it is necessary that κ_2 is much larger than the damping rates of the other modes. This suggests to adiabatically eliminate this mode, an approximation that leads to the simplified master equation [8–10]

$$\dot{\rho} = -i[H_c, \rho] + \kappa_0 \mathcal{D}[a_0]\rho + \kappa_1(N+1)\mathcal{D}[a_1]\rho + \kappa_1 N \mathcal{D}[a_1^\dagger]\rho + \Gamma \mathcal{D}[a_0^\dagger a_1^2]\rho. \quad (5)$$

Equation (5) is written in the interaction picture with respect to $H_0 = \omega_0 a_0^\dagger a_0 + \omega_1 a_1^\dagger a_1$, and the reduced density matrix ρ is $\rho = \text{Tr}_{\text{mode } 2}[W]$. The reduced collision Hamiltonian H_c is

$$H_c = V_{0000} a_0^\dagger a_0^\dagger a_0 a_0 + V_{0101} a_0^\dagger a_1^\dagger a_0 a_1 + V_{1111} a_1^\dagger a_1^\dagger a_1 a_1, \quad (6)$$

and $\Gamma = 4|V_{0211}|^2/\kappa$. Consistently with Ref. [9] we call the limiting cases $\Gamma \ll \kappa_0$ and $\Gamma \gg \kappa_0$ the weak and strong collision regimes, respectively. The reduced master equation (5) forms the basis of most previous studies of binary collision atom lasers.

The master equations (1) and (5) can easily be solved numerically using standard quantum Monte Carlo simulation techniques [24,25]. This is discussed in detail in Ref. [12], and we only recall that these simulations are significantly facilitated by the fact that between quantum jumps the total number of atoms is conserved. In this way one has to propagate only a small number of actually populated states, a situation evidently advantageous with regard to both memory and CPU time requirements².

III. ELASTIC COLLISIONS AND LASER LINEWIDTH

A. Linearized fluctuation analysis for the two-mode system

In order to obtain an analytical approximation for the laser linewidth in the two-mode system a linearized

fluctuation analysis can be performed [32,33]. To this end the master equation (5) is converted to a Fokker-Planck equation using the P -function representation as described in [9]. This equation can be transformed to polar coordinates $\alpha_j = \sqrt{n_j} e^{i\phi_j}$, where α_j denotes the complex amplitudes originally appearing in the Fokker-Planck equation [34,35]. This leads to stochastic differential equations

$$dn_0 = [\Gamma n_1^2(n_0 + 1) - \kappa_0 n_0]dt + dS_{n_0}, \quad (7)$$

$$dn_1 = [\kappa_1(N - n_1) - 2\Gamma n_1^2(n_0 + 1)]dt + dS_{n_1}, \quad (8)$$

$$d\phi_0 = [-V_{0000}(2n_0 - 1) - V_{0101}n_1]dt + dS_{\phi_0}, \quad (9)$$

$$d\phi_1 = [-V_{1111}(2n_1 - 1) - V_{0101}n_0]dt + dS_{\phi_1}. \quad (10)$$

The correlation matrix for the stochastic forces $d\mathbf{S}^T = (dS_{n_0}, dS_{n_1}, dS_{\phi_0}, dS_{\phi_1})$ is given by

$$\mathbf{D} = \begin{pmatrix} 2\Gamma n_1^2 n_0 & -2\Gamma n_1^2 n_0 & -2V_{0000}n_0 & -V_{0101}n_0 \\ -2\Gamma n_1^2 n_0 & 2\kappa_1 N n_1 - 2\Gamma n_1^2 n_0 & -V_{0101}n_1 & -2V_{1111}n_1 \\ -2V_{0000}n_0 & -V_{0101}n_1 & \Gamma n_1^2/(2n_0) & \Gamma n_1/2 \\ -V_{0101}n_0 & -2V_{1111}n_1 & \Gamma n_1/2 & \frac{\kappa_1 N}{2n_1} + \frac{\Gamma n_0}{2} \end{pmatrix}. \quad (11)$$

In the limit $n_0 \gg 1$ one obtains from Eqs. (7) and (8) the above-threshold semiclassical steady-state populations [9]

$$\bar{n}_0 = \frac{1}{2} \frac{\kappa_1}{\kappa_0} (N - \bar{n}_1), \quad (12)$$

$$\bar{n}_1 = \sqrt{\frac{\kappa_0}{\Gamma}}, \quad (13)$$

the threshold condition being $N > \sqrt{\kappa_0/\Gamma}$. The relation (12) between \bar{n}_0 and \bar{n}_1 also holds in the full quantum-mechanical two- and three-mode models [12]. To proceed further we introduce the fluctuation variables $\delta n_j = n_j - \bar{n}_j$ and $\delta \phi_j = \phi_j - \bar{\phi}_j$, where the phase drift variables $\bar{\phi}_j$ obey the deterministic equations obtained from Eqs. (9) and (10) by discarding the stochastic forces and substituting \bar{n}_j for n_j . In the linear approximation the time evolution of the fluctuations $\delta \mathbf{A}^T = (\delta n_0, \delta n_1, \delta \phi_0, \delta \phi_1)$ is given by

$$d\delta \mathbf{A} = -\mathbf{k} \delta \mathbf{A} dt + d\mathbf{S} \quad (14)$$

where the matrix \mathbf{k} is obtained by linearizing the drift terms in Eqs. (7) – (10) around the steady-state values \bar{n}_j . The correlation matrix for the stochastic forces $d\mathbf{S}$ is given by the matrix \mathbf{D} of Eq. (11) after replacing n_j by \bar{n}_j . Supposing that $\delta \mathbf{A} = \mathbf{0}$ at time $\tau = 0$, the probability distribution of the fluctuations at a later time τ is given by [34]

$$p(\delta \mathbf{A}, \tau) = [(2\pi)^4 \det \boldsymbol{\sigma}(\tau)]^{-1/2} \exp[-\frac{1}{2} \delta \mathbf{A}^T \boldsymbol{\sigma}^{-1}(\tau) \delta \mathbf{A}] \quad (15)$$

with

²In particular, in the calculation of correlation functions of the type $\langle a_j^\dagger(t+\tau) a_j(t) \rangle$, one normally has to evolve four wave functions $[1 \pm (i)a_j]|\Phi(t)\rangle$ for a given initial sample wave function $|\Phi(t)\rangle$ if one uses the method of Ref. [25]. However, due to the conservation of atom number in this case the contributions from $|\Phi(t)\rangle$ and $a_j|\Phi(t)\rangle$ in the time development of $[1 + a_j]|\Phi(t)\rangle$ can be distinguished. This means that one obtains a contribution to the correlation function even from a single propagation. As it is more advantageous to average over a large number of different $|\Phi(t)\rangle$ than to use many simulations for a small number of initial wave functions one obtains a significant additional decrease in computation time.

$$\sigma(\tau) = \int_0^\tau d\tau' \exp[-\mathbf{k}(\tau - \tau')] \mathbf{D} \exp[-\mathbf{k}^T(\tau - \tau')]. \quad (16)$$

Assuming as usual that the first-order correlation function $C_0(\tau) = \langle a_0^\dagger(\tau) a_0(0) \rangle$ is determined only by the phase fluctuations one obtains

$$\begin{aligned} C_0(\tau) &= \int d^4\delta\mathbf{A} \bar{n}_0 e^{-i\phi_0} p(\delta\mathbf{A}, \tau) \\ &= \bar{n}_0 e^{-i\bar{\phi}_0} \exp[-\frac{1}{2}\sigma_{33}(\tau)], \end{aligned} \quad (17)$$

where the deterministic phase drift is also included. This phase drift leads to a shift of the center of the power spectrum (the Fourier transform of the correlation function) by an amount $2V_{0000}\bar{n}_0 + V_{0101}\bar{n}_1$ with respect to the collisionless case. The behavior of the correlation function $C_0(\tau)$ is thus essentially determined by the covariance matrix element $\sigma_{33}(\tau)$.

The right hand side of Eqs. (7) – (10) depends only on the atom numbers n_0 and n_1 ³. This means that in that two-mode scheme the phase diffusion due to interatomic collisions is induced solely by atom number fluctuations. This observation significantly simplifies the further analytical treatment as it leads to the matrix \mathbf{k} having two vanishing column vectors. Hence the covariance matrix $\sigma(\tau)$ can be evaluated explicitly in a straightforward way according to Eq. (16). However, the ensuing expression is still rather complicated, so that in the following we restrict the discussion to two limiting cases which illustrate the essential aspects of the influence of the elastic collisions on the laser linewidth. It should also be noted at this point that in the two-mode system $C_0(\tau)$ does not depend on elastic collisions between pumping mode atoms, which are characterized by the parameter V_{1111} .

(a) $V_{0101} = 0$. Expanding the expression for $\sigma_{33}(\tau)$ to leading order in the parameter \bar{n}_0 one obtains

$$\begin{aligned} \sigma_{33}(\tau) &= \tau[w + \kappa_0/(2\bar{n}_0)] \\ &\quad + \frac{[1 - \exp(-q\tau)]}{q} (8V_{0000}^2\bar{n}_0/q - 2w) \\ &\quad + \frac{[1 - \exp(-2q\tau)]}{2q} (w - 8V_{0000}^2\bar{n}_0/q) \end{aligned} \quad (18)$$

with

$$w = V_{0000}^2 \frac{\kappa_1 N}{\kappa_0^2} \left(2 + 2\sqrt{\kappa_0/\Gamma} + \frac{N}{N - \sqrt{\kappa_0/\Gamma}} \right) \quad (19)$$

and

³This is in contrast to the three-mode system, where an explicit phase dependence is introduced through the coherent pumping term $V_{0211}a_0^\dagger a_2^\dagger a_1 a_1 + h.c.$

$$q = \frac{4\kappa_0^2}{4\kappa_0 + \kappa_1\bar{n}_1/\bar{n}_0}. \quad (20)$$

From Eq. (12) it follows that $\kappa_1 N$ and thus w are of the order of \bar{n}_0 . In the first line of Eq. (18) the term $\kappa_0/(2\bar{n}_0)\tau$ was included although it is not of the same order in \bar{n}_0 as the other terms. This is because it describes the behavior of the correlation function if V_{0000} is very small (cf. Fig. 3). The parameter q of Eq. (20) is one of the eigenvalues of the upper left 2×2 -minor of the matrix \mathbf{k} . It can thus be interpreted as the inverse of one of the timescales relevant in the dynamics of the atom number fluctuations. The other eigenvalue is of the order of \bar{n}_0 and introduces a much shorter time scale. It does not play a significant role for the characterization of the correlation function.

Equation (17) implies that the most important aspects of the correlation function can be inferred from the study of the behavior of $\sigma_{33}(\tau)$ in the time interval where it is smaller than or of the order of unity. From Eq. (18) two different kinds of behavior can thus be distinguished, depending on the magnitude of V_{0000} . As long as $\sigma_{33}(\tau = 1/q) \ll 1$ the time evolution of $\sigma_{33}(\tau)$ relevant for $C_0(\tau)$ is well approximated by

$$\sigma_{33}(\tau) \simeq \tau[w + \kappa_0/(2\bar{n}_0)]. \quad (21)$$

This is because as time increases the second and the third term on the right-hand side of Eq. (18) remain constant for $\tau > 1/q$, while the first term increases. For such values of V_{0000} $C_0(\tau)$ decays therefore exponentially, and the power spectrum is Lorentzian. If $w \gg \kappa_0/(2\bar{n}_0)$ the linewidth is proportional to $V_{0000}^2\bar{n}_0$. In contrast to the situation with conventional lasers, it *increases* linearly with the number of atoms in the laser mode.

In case $\sigma_{33}(\tau = 1/q) \gg 1$ the decay of the correlation function can accurately be approximated by expanding the exponentials in Eq. (18) up to second order. This yields the expression

$$\sigma_{33}(\tau) \simeq 4V_{0000}^2\bar{n}_0\tau^2. \quad (22)$$

Under these circumstances the correlation function decays like a Gaussian. The spectrum is thus itself a Gaussian and its linewidth is proportional to $V_{0000}\sqrt{\bar{n}_0}$. The atom laser linewidth still increases with \bar{n}_0 , albeit less dramatically than in the preceding case.

Figure 3 suggests that a rough estimate of the value of V_{0000} at which the decay of the correlation function changes from exponential to Gaussian can be obtained in the following way. One determines V_{0000} such that both Eqs. (21) and (22) yield the same solution for the condition $\sigma_{33}(\tau) = 2/\ln 2$. In this way one obtains as “critical” value

$$V_{0000, crit} = \sqrt{\frac{8}{\ln 2}} \frac{\sqrt{\bar{n}_0}}{w'} \quad (23)$$

with $w' = w/V_{0000}^2$. If \bar{n}_0 is increased the change thus occurs for smaller values of V_{0000} .

It is instructive to compare these results to those obtained for a damped harmonic oscillator with Hamiltonian $H = \omega_0 a_0^\dagger a_0 + V_{0000} a_0^\dagger a_0^\dagger a_0 a_0$. Denoting in analogy with Eq. (1) the damping coefficient by κ_0 and the external pumping strength by N , a linearized fluctuations analysis yields for the relevant covariance matrix element

$$\begin{aligned} \bar{\sigma}(\tau) = \tau & \left[8V_{0000}^2 \frac{N^2 + N}{\kappa_0} + \kappa_0/2 \right] \\ & - 16V_{0000}^2 \frac{N^2 + N/2}{\kappa_0^2} [1 - \exp(-\kappa_0\tau)] \\ & + 4V_{0000}^2 \frac{N^2}{\kappa_0^2} [1 - \exp(-2\kappa_0\tau)], \end{aligned} \quad (24)$$

where no further approximation has been made, in contrast to the case of Eq. (18). In the limit of large V_{0000} Eq. (24) yields $\bar{\sigma}(\tau) = 4V_{0000}^2 N \tau^2$, a result formally identical to Eq. (22) since the equilibrium population \bar{n}_0 becomes N in the one-mode system. Such a connection between the linewidths for this one-mode system and the atom laser has been previously noted in Ref. [23]. Despite their formal similarity, there are however significant differences between Eqs. (18) and (24). First, there is no obvious correspondence between the coefficients multiplying their respective time-dependent contributions, and they are not of the same order in \bar{n}_0 . In addition, the one-mode system is characterized by a very large “fundamental” linewidth $\kappa_0/2$ which masks a possible quadratic dependence on V_{0000} . Finally, the above-mentioned limiting behavior is reached for values of V_{0000} much larger than in the two-mode atom laser system.

(b) $V_{0000} = 0$. In this case the expansion of $\sigma_{33}(\tau)$ to leading order in \bar{n}_0 yields

$$\sigma_{33}(\tau) = \left(\frac{V_{0101}^2}{2\Gamma\bar{n}_0} + \frac{\kappa_0}{2\bar{n}_0} \right) \tau \quad (25)$$

The correlation function thus decays exponentially for all values of V_{0101} . A qualitative change in behavior as in the previous situation does not occur. The linewidth of the spectrum is now proportional to V_{0101}^2/\bar{n}_0 , and becomes *narrower* when the population of the laser mode is increased, very much like the familiar Shawlow-Townes linewidth of conventional lasers.

The analytical results described so far were compared to numerical calculations of the correlation function. Both the weak-collision regime and the strong-collision regime were investigated for the explicit values $\Gamma/\kappa_0 = 1/15$ and 15 and $\kappa_1/\kappa_0 = 10$ and 100 . For $V_{0101} = 0$ a good agreement could generally be observed between the numerical result and the analytical approximation based on Eq. (17). A representative example is shown in Fig. 2 where the parameter values are $\Gamma/\kappa_0 = 15$, $\kappa_1/\kappa_0 = 100$, $N = 1.85$, $V_{0000}/\kappa_0 = 2.5$, $V_{0101} = 0$. The numerical results are well approximated by a Gaussian. Considerable deviations occurred only in the case $\Gamma/\kappa_0 = 1/15$, $\kappa_1/\kappa_0 = 100$. For these values the quantum effects in the

population dynamics are particularly large, e.g. choosing N such that $\bar{n}_0 = 50$ according to Eq. (12) led to a numerical value of 100 whereas in the other case deviations were only a few percent. It should be noted that in the two-mode system the population dynamics do not depend on the interatomic collisions.

The transition between the exponential and Gaussian regimes in the behavior of the correlation function $C_0(\tau)$ is illustrated in Fig. 3. It shows the inverse of the “half-life time” $\tau_{1/2}$ of $C_0(\tau)$ (defined by $\sigma_{33}(\tau_{1/2}) = \ln 2/2$) as a function of V_{0000} for the parameters $\Gamma/\kappa_0 = 15$, $\kappa_1/\kappa_0 = 10$, and $N = 20.3$, corresponding to $\bar{n}_0 = 100$, on a doubly-logarithmic scale. The full curve depicts $\tau_{1/2}^{-1}$ as determined from Eq. (17), the dashed and the dotted curves are obtained from the approximate relations (21) and (22), respectively. The results of numerical Monte Carlo simulations are shown as circles (\bullet). The figure indicates that the analytical predictions, in particular the existence of two different regimes and the transition between them, are accurately confirmed by the numerical calculations. The approximations based on Eqs. (21) and (22) are almost indistinguishable from the full analytical expression in their respective regions of validity.

The results of a similar study for the collision coefficient V_{0101} are shown in Fig. 4. There, the dashed curve is derived from the approximation (25). The global behavior of the numerical results is well approximated by the analytical description, in particular the quadratic dependence of the half-life time on V_{0101} is recovered. However, the analytical prediction typically overestimates the numerical half-life time by a factor of two. This behavior is also observed in examples with other parameter values. This indicates that the atom number fluctuations in the pump mode are not as well described by a linearized ansatz as the fluctuations in the laser mode.

In case both collision coefficients V_{0000} and V_{0101} are non-vanishing the analytical form of a reliable approximation to Eq. (16) would be rather complicated. However, it can be seen that all terms involving V_{0101} are at most of the order of \bar{n}_0^0 . Taking into account Eq. (23) this means that in the limit of large \bar{n}_0 the function $\sigma_{33}(\tau)$ can eventually be approximated by Eq. (22). The comparison with numerical results shows that Eq. (17) is still accurate when both collision coefficients are present. The calculations indicate that the correlation time decreases, i.e., the linewidth broadens, if the second collision coefficient is switched on in addition to the first. If the values of the collision coefficients are such that one of them has a much larger impact on the correlation time than the other – which is checked by comparing Eqs. (18) and (25) – then the approximate results of the limiting cases (a) and (b) still hold. If both have roughly the same impact then the ensuing correlation time will still be of the order of the result inferred from the approximations.

B. Operating regimes of the atom laser

In the following we apply these results to the atom laser scheme of Refs. [8,11,12]. In this model the collision mechanism is the laser-induced dipole-dipole interaction. Due to energy and momentum conservation, the associated selection rules allow certain collision coefficients, most notably V_{0000} , to become small in comparison with V_{0211} . In our numerical example we examine a situation where only the elastic collision coefficients $V_{0000} = V_{0211}/15$ and $V_{0101} = V_{0211}$ are taken into account. A further interesting feature of this system is the fact that the dominant collision mechanism and the atom cavity are realized with laser fields whose intensities can be varied independently from each other. This means that while the ratio V_{ijkl}/V_{0211} is virtually fixed once the experimental setup has been optimized there is still considerable latitude in the choice of the relative magnitude of V_{0211} compared to κ_0 .

For the numerical simulations involving the three-mode model to remain tractable, we chose parameters such that the laser linewidth is of the order of the natural linewidth κ_0 and the equilibrium laser mode population of the order of $\bar{n}_0 = 50$. With the help of Eq. (22) one estimates from the condition $\sigma_{33}(2/\kappa_0) = 1$ that $V_{0000} = 0.035\kappa_0$. This also fixes $V_{0101} = V_{0211} = 15V_{0000}$. From Eq. (25) one obtains $\Gamma = 5.63 \times 10^{-3}\kappa_0$ as a suitable value for the pumping parameter by again using the condition $\sigma_{33}(2/\kappa_0) = 1$. Choosing $\kappa_1 = 20\kappa_0$ yields $N = 18.3$. In this way all parameters which enter the two-mode model are determined. The relative orders of magnitude appear to be reasonable. The numerical calculation yields a half-life time of about $0.5\kappa_0$ and $\bar{n}_0 = 68$ which agrees reasonable well with our stated objective.

In order to put this result into the right perspective it is necessary to perform corresponding calculations for the three-mode model also. This is because the two-mode scheme assumes that $V_{0211}, \kappa_2 \rightarrow \infty$ with the ratio $\Gamma = 4V_{0211}^2/\kappa_2$ held fixed, without taking into account the need to assign finite values to the ratios V_{ijkl}/V_{0211} .

One of the most important results of a systematic comparison between the predictions of the two-mode and the three-mode models is that in the presence of elastic collisions the equilibrium populations may react at least as sensitively to the effects of “non-adiabaticity” as the linewidth. In other words, as the values of V_{0211} and κ_2 are lowered with Γ fixed, \bar{n}_0 can drop significantly below its adiabatic value even before the linewidth is considerably affected. This occurs as soon as the value of V_{0211} becomes of the order of the other collision coefficients. This may appear surprising in view of the fact that in the two-mode system the population dynamics are completely independent of the elastic collisions. A possible explanation is that the levels $|n_0, n_1, n_2\rangle$ and $|n_0 + 1, n_1 - 2, n_2 + 1\rangle$ which are coupled by the coherent pumping experience an effective detuning due to the collisions. Hence, the predictions of the

two-mode system for a given set of parameters should always be compared the full three-mode simulations to assess their reliability. In the present example, however, the population decrease in the three-mode system (with $\kappa_2 = 4V_{0211}^2/\Gamma = 201\kappa_0$) is not very pronounced ($\bar{n}_0 = 48$) while the linewidth hardly changes compared to its two-mode value. These results indicate that in the atom laser scheme of Refs. [8,11,12] it should be possible to obtain a linewidth of the order of or smaller than the natural linewidth κ_0 of the laser mode.

IV. RELATIVE PHASE BETWEEN TWO ATOM LASERS

The question of how a definite relative phase can be established between two independently created, interfering Bose condensates has been the subject of intensive theoretical study in the recent past [26–31]. It was shown that one needs not resort to the concept of spontaneously broken symmetry to model the creation of interference patterns. Rather, these structures are brought about by the very act of observation: A sequence of measurements leads to the creation of an entangled state of the condensates pair which can be assigned a relative phase. However, this phase varies randomly from one realization of the experiment to the other.

In this section we examine whether such a relative phase is also created when the output beams of two independent atom lasers are brought to interference. The purpose of this study is to determine to which extent the concept of measurements-induced phase built-up retains its validity also in open systems. The entanglement between the atoms in the two laser modes is now influenced by the continuing pumping processes, and it is natural to ask if a well-defined relative phase can persist and what its properties are, e.g. with respect to diffusion. The question is also of interest with regard to related studies of optical systems [36].

We consider a gedanken experiment where the outputs of two two-mode atom lasers are brought to interference with the help of a 50:50 atomic beam splitter⁴. In the spirit of Ref. [30], the effective non-hermitian Hamiltonian describing the Schrödinger-like evolution of the two-laser system is

$$H_{eff} = H_a + H_b \quad (26)$$

where

⁴The use of a two-mode rather than a three-mode laser model simplifies the numerics while retaining the essential physical characteristics of the binary collision model. Furthermore, in the Monte Carlo wave function of a single two-mode atom laser a single number state is populated, so that it closely resembles the state of an isolated Bose condensate.

$$H_a = H_c - \frac{i}{2}[\kappa_0 a_0^\dagger a_0 + \kappa_1(N+1)a_1^\dagger a_1 + \kappa_1 a_1 a_1^\dagger + \Gamma a_1^{\dagger 2} a_0 a_0^\dagger a_1^2] \quad (27)$$

is the effective Hamiltonian for the first atom laser. The Hamiltonian for the second laser is defined similarly, but using annihilation operators b_i . The definition of the various quantum jump operators is obvious from Eqs. (26) and (27) except for those which describe atoms leaving the laser modes after passing through the beam splitter. They are given by

$$c_\pm = (a_0 \pm b_0)/\sqrt{2}. \quad (28)$$

It is a simple matter to perform quantum Monte Carlo simulations within this framework. The general form of the simulation wave function can be written as

$$|\Psi(t)\rangle = \sum_{n_{01}=n_{01,min}}^{n_{01,max}} c_{n_{01}}(t) |n_{01}, n_{11}, n_{02} = N_L - n_{01}, n_{12}\rangle, \quad (29)$$

where n_{ij} denotes the number of atoms in mode i of laser j and N_L is the total number of atoms in the two laser modes. The distribution function of the relative phase between the laser modes is conveniently expressed in terms of the overcomplete set of phase states [30]

$$|\Phi\rangle_{N_L} = \frac{1}{\sqrt{2^{N_L} N_L!}} (a_0^\dagger e^{i\Phi/2} + b_0^\dagger e^{-i\Phi/2})^{N_L} |0, 0, 0, 0\rangle. \quad (30)$$

Because of the relation

$$(a_0 + b_0)|\Phi\rangle_{N_L} = \sqrt{2N_L} \cos(\Phi/2) |\Phi\rangle_{N_L} \quad (31)$$

one can think of the state $|\Phi\rangle_{N_L}$ as possessing a definite relative phase Φ between the laser modes within the context of the interference experiment. The wave function $|\Psi(t)\rangle$ can be expanded on these phase states as

$$|\Psi(t)\rangle = \int_{-\pi}^{\pi} \frac{d\Phi}{2\pi} c(\Phi, t) |\Phi\rangle_{N_L} \quad (32)$$

where the phase distribution amplitude $c(\Phi, t)$ is given by

$$c(\Phi, t) = 2^{N_L/2} \sum_{n_{01}=n_{01,min}}^{n_{01,max}} \left(\frac{n_{01}!(N_L - n_{01})!}{N_L!} \right)^{1/2} e^{i(N_L/2 - n_{01})\Phi} c_{n_{01}}(t). \quad (33)$$

Several physical processes influence the phase distribution: they are (i) the loss processes induced by the action of the operators c_\pm leading to the entanglement of the atom lasers; (ii) the pumping into the laser modes described by the action of a_0^\dagger and b_0^\dagger ; (iii) the collision interactions taking place between quantum jumps and

described by H_c ; and (iv) the non-Hermitian time evolution described by the Γ -term in Eq. (27). Whereas (i) and (iii) are also present in the case of Bose condensates, the influence of (ii) and (iv) occurs only for the laser. It is not a priori evident that the distribution function $c(\Phi, t)$ still yields a well-defined phase in the presence of these terms.

Figure 5 shows typical phase distributions $|c(\Phi)|^2$ obtained from Monte Carlo wave function simulations of the two-laser problem. All examples in this section are calculated for the parameters $\Gamma = 21.9\kappa_0$, $\kappa_1 = 54.6\kappa_0$ and $N = 2.2$. Figure 5(a) illustrates the collisionless case $V_{0000} = 0$. Obviously, in this case the state of the system can be ascribed a well-defined phase. The distribution is symmetric with respect to $\Phi = 0$, reflecting the fact that the method of detection does not discriminate between the states $|\Phi\rangle$ and $|\Phi + \pi\rangle$. However, it is much broader than for the corresponding phase state $|\Phi\rangle_{N_L}$ shown in Fig. 5(b) for comparison. That means that in general an approximation of the instantaneous Monte Carlo wave function by the superposition of two phase states is not appropriate.

The influence of elastic collisions on the phase distribution is demonstrated in Fig. 5(c). There, the value $V_{0000} = 0.25\kappa_0$ was chosen, yielding a half-life time of $\tau_{1/2} = 0.33/\kappa_0$ for the correlation function of the single atom laser, a value much smaller than the value $\tau_{1/2} = 142/\kappa_0$ for the collisionless case. Figure 5(c) shows that the collisions tend to broaden the phase distribution. Furthermore the symmetry of the distribution is lost with respect to both the height and the center of the two maxima. Nonetheless it is still reasonable to associate a well-defined phase with this state.

From a formal point of view it is not too surprising that the Monte Carlo wave functions retain their definite-phase character in the presence of the pump processes. In the entangled state (29) which is created by the interference at the beam splitter the coefficients $c_{n_{01}}(t)$ of significant weight are concentrated in a small interval around $n_{01} = N_L/2$. That means that the processes (ii) and (iv) change the wave function slightly, but without altering its basic character.

Having established that a well-defined phase still exists in the presence of pump and collisions, we now turn to its dynamical properties, in particular its diffusive behavior. To this end we associate an instantaneous phase with the Monte Carlo wave function $|\Psi(t)\rangle$ via the relation

$$\cos^2[\Phi(t)/2] = \frac{\langle \Psi(t) | (a_0^\dagger + b_0^\dagger) ((a_0 + b_0) | \Psi(t) \rangle)}{2 \langle \Psi(t) | a_0^\dagger a_0 + b_0^\dagger b_0 | \Psi(t) \rangle}, \quad (34)$$

compare with Eq. (31). Note that Eq. (34) accounts for the fact that the phase is only well-defined for the interval between 0 and π . Note also that the use of other equivalent definitions of the relative phase, e. g. the maximum of the phase distribution $|c(\Phi)|^2$, is possible. Eq. (34) allows one to numerically evaluate averages of the form

$$A_{\bar{\Phi}}(t) = \langle \cos^2[\Phi(t)/2] \rangle_{\Phi(t=0)=\bar{\Phi}}. \quad (35)$$

which measures the diffusion of an ensemble of Monte Carlo wave functions which all have the same initial relative phase $\tilde{\Phi}$. One can also determine the variance

$$V_{\tilde{\Phi}}(t) = \langle \cos^4[\Phi(t)/2] \rangle_{\Phi(t=0)=\tilde{\Phi}} - A_{\tilde{\Phi}}^2(t), \quad (36)$$

where $\cos^4[\Phi(t)/2]$ is calculated as the square of $\cos^2[\Phi(t)/2]$ as inferred from Eq. (34). A third quantity of interest is

$$M(t) = \langle \langle \{ \cos^2[\Phi(t)/2] - \cos^2[\Phi(t=0)/2] \}^2 \rangle \rangle, \quad (37)$$

where the inner set of brackets indicates an average over quantum trajectories with the same initial $\Phi(t=0)$ and the outer one averaging over the different values of $\Phi(t=0)$.

The quantities introduced in Eqs. (35) – (37) allow one to study the diffusion properties of the relative phase as defined by Eq. (34). If this definition is meaningful then there should be a connection between this result and the phase dynamics of an individual atom laser which, as we have seen in Sec. III, is well described within the framework of the linearized fluctuation approach. In particular, the probability distribution for the phase ϕ_0 of a single laser is given to a good approximation by

$$P(\phi_0, t) = \frac{1}{\sqrt{2\pi\sigma_{33}(t)}} \exp \left\{ -(\phi_0 - \bar{\phi}_0)^2 / [2\sigma_{33}(t)] \right\} \quad (38)$$

where $\bar{\phi}_0$ is the laser phase at $t=0$ and $\sigma_{33}(t)$ is given by Eq. (16).

Eq. (38) predicts that the time dependence of the quantities defined in Eqs. (35) – (37) are of the form

$$\begin{aligned} A_{\tilde{\Phi}}(t) &= \frac{1}{2} \left\{ 1 + \exp[-\sigma_{33}(t)] \cos \tilde{\Phi} \right\}, \\ V_{\tilde{\Phi}}(t) &= \frac{1}{8} \{ 1 - 2 \exp[-2\sigma_{33}(t)] \cos^2 \tilde{\Phi} + \\ &\quad \exp[-4\sigma_{33}(t)] \cos 2\tilde{\Phi} \}, \\ M(t) &= \frac{1}{4} \{ 1 - \exp[-\sigma_{33}(t)] \}. \end{aligned} \quad (39)$$

where the calculation of $M(t)$ assumes that all initial phase differences have equal probability.

Fig. 6 compares these predictions to a Monte Carlo evaluation of Eqs. (35) – (37). Fig. 6(a) and (b) show $A_0(t)$, $V_0(t)$ and $V_{\pi/2}(t)$ for the collisionless case and the other parameters as in Fig. 5. Figure 6(c) depicts $M(t)$ for $V_{0000} = 0.25\kappa_0$. In all cases the behavior of the numerical results is well described by the expressions (39). A similar degree of agreement was also found in further examples. These results indicate that it is indeed possible to associate a definite relative phase with single simulations of the interference experiment and that this heuristically introduced phase behaves as predicted by the description of the atom laser in terms of stochastic processes [35].

Experimentally, the phase can be observed (as soon as the correlation time is much larger than the mean time delay between atom emissions) by monitoring the ratio

of the output intensities at the two beam splitter ports. Thus, the notion of spontaneously broken symmetry appears not to be the most natural concept to explain these experiments, either.

As to the numerical simulations it should be remarked that $\Phi(t)$ as defined through Eq. (34) never assumes the values 0° or 180° . Instead, it always remains in the region between approximately 10° and 170° , for the collisionless case (and the specific parameters of our simulations), and a somewhat narrower interval in the presence of collisions. In fact, the numerical calculations of $A_0(t)$ and $V_0(t)$ were performed using the actual minimum value of Φ as initial value. For this value $\cos^2(\Phi/2)$ is still close to 1. At first sight, this observation might invalidate the description of the phase dynamics in terms of an unbounded stochastic process. However, this seems rather to be a shortcoming of the simple definition (34). It is more appropriate to think of the phase as having at every instant a well-localized but not infinitely narrow distribution. Equation (34) assigns a definite value to this distribution but may become inaccurate at the extreme values of $\cos^2[\Phi(t)/2]$. In the calculations, this is compensated for by the fact that the probability that $\cos^2[\Phi(t)/2]$ assumes its actual minimum or maximum is somewhat larger than the probability for intermediate values. Furthermore, it should be noted that in Fig. 6(c) the numerical result assumes quite exactly the predicted long-time value of 0.25. This justifies thinking of the probability distribution of the relative phase as being approximately constant.

V. SUMMARY AND CONCLUSIONS

Using a linearized fluctuation analysis we have obtained analytical estimates for the influence of elastic collisions on the linewidth of a two-mode atom laser system. In case elastic interactions between laser mode atoms provide the dominant collision mechanism the power spectrum is Gaussian and the linewidth scales as $V_{0000}\sqrt{\bar{n}_0}$ for weak enough collisions. It undergoes a transition to a Lorentzian, with a linewidth proportional to \bar{n}_0 , as V_{0000} increases. A Schawlow-Townes-like behavior of the linewidth is recovered if collisions between pump and laser mode atoms are dominant. In that case the lineshape is Lorentzian with a linewidth proportional to V_{0101}^2/\bar{n}_0 . The accuracy of the analytical predictions was confirmed by Monte Carlo wave function simulations. In addition, it was shown in a numerical study that the recently proposed atom laser scheme of Refs. [8,11,12] might be capable of producing linewidths of the order of or smaller than the natural linewidth κ_0 of the laser mode. In the course of the numerical study the usefulness of the analytical estimates in the search for suitable operating regimes became apparent.

In a second aspect of the phase dynamics, we investigated the relative phase between two interfering atom

lasers. This study extends earlier work on the relative phase of Bose condensates to the realm of open systems. The analysis of suitably defined phase distribution functions showed that a physically meaningful relative phase can be ascribed to single runs of an interference experiment. A close connection with the phase dynamics of the single atom laser, as described by a Fokker-Planck equation, was established by studying the diffusion properties of this relative phase.

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FIG. 1. Schematic three-mode atom laser scheme.

FIG. 2. Comparison between numerical calculation of the correlation function (full curve) and the result of Eq. (17) (dashed) for parameter values $\Gamma/\kappa_0 = 15$, $\kappa_1/\kappa_0 = 100$, $N = 1.85$, and $V_{0000}/\kappa_0 = 2.5$, $V_{0101} = 0$.

FIG. 3. Inverse half-life time $\tau_{1/2}^{-1}$ of the correlation function as a function of V_{0000} for parameter values $\Gamma/\kappa_0 = 15$, $\kappa_1/\kappa_0 = 10$, $N = 20.3$ and $V_{0101} = 0$. Shown are the results of Eq. (17) (full curve), Eq. (21) (dashed), Eq. (22) (dotted), and numerical calculations (\bullet).

FIG. 4. Inverse half-life time $\tau_{1/2}^{-1}$ of the correlation function as a function of V_{0101} for parameter values $\Gamma/\kappa_0 = 15$, $\kappa_1/\kappa_0 = 10$, $N = 20.3$ and $V_{0000} = 0$. Shown are the results of Eq. (17) (full curve), Eq. (25) (dashed), and numerical calculations (\bullet).

FIG. 5. Examples of phase distributions $|c(\Phi)|^2$: (a) typical atom laser state for the collisionless case (other parameters as given in the text); (b) phase state as defined by Eq. (30) with $\Phi = 45^\circ$ and $N_L = 120$; (c) typical atom laser state in the presence of collisions ($V_{0000} = 0.25\kappa_0$).

FIG. 6. Comparison between numerical calculation (full curves, averages over approx. 250 simulations each) and analytical prediction (dashed) according to Eqs. (39) for: (a) $A_0(t)$ with $V_{0000} = 0$; (b) $V_0(t)$ and $V_{\pi/2}(t)$ with $V_{0000} = 0$; (c) $M(t)$ with $V_{0000} = 0.25\kappa_0$. Other parameters as given in the text.











